Supplementary notes: see web page

Some slides today based on Nils Andersens’ (from 2004)
PRACTICAL DETAILS ABOUT THE COURSE


- The book has a [WWW tutor page](#).

- Mondays: lecture 09:15-11:00

- Wednesdays 09:15-12:00:
  - 2 hours lecture,
  - 1 hour discussion, eg exercises

- **Exercises** assigned Monday and/or Wednesday, due following Monday (not optional, 5 out of 7 sets must be accepted for 7.5 ECTS points course credit)

- Final exam optional, if you wish credit with a grade on the 13-scale. Week 44 (31 October or 2 November)
Why learn logic?

History of logic

Formal logic as a science

The three worlds:
- The real world
- Proofs and theorems
- Models and validity

Today: begin Propositional logic
- Sentences
- Natural deduction
 WHY LEARN LOGIC ?

Correctness of constructed systems: telephones, computers (eg CPUs, floating point, cache, etc.), control systems for autos (e.g., ABS, emissions), factories, power plants (including nuclear), computer graphics, etc, etc, etc.

▶ Traditional approach: debugging. (Build it first, then test, then release.)
▶ Problem: too late, e.g., correcting an already-constructed system can be
  • expensive (eg, the Pentium floating-point unit fiasco) or
  • impossible (eg, an auto ABS system, airplane control, nuclear plant)
▶ Prevention (of bugs) is better than cure

Applied logic is showing its worth for describing, building and analysing complex systems (both hardware and software).

Many people are employed in applied logic, particularly in England, Germany, France and the United States.
ROLES OF LOGIC

Descriptive

- Propositional logic, e.g., circuits
  Values $T, F$ or $0, 1$
- Predicate logic, e.g., theorems
  Values from a set, e.g., $\mathbb{N}$
- Temporal logic, e.g., protocols, systems
  Mainly control flow

Analysis

- Test freedom from errors, race conditions, ...
- Test other well-behavedness, e.g., of code from WWW
- Test correctness with respect to specifications
- Test equivalence, e.g., of circuits

Synthesis

- Transformation: Specification $\Rightarrow$ Implementation
- Formal reasoning, proofs
LOGIC IN ANTIQUITY

The science of inferring correctly. Some conclusions only depend on the form of the argument and not on the actual contents.

Doesn’t deal with how humans think (psychology) or whether the statements actually agree with facts (theory of knowledge).


All $M$ are $P$

All $S$ are $M$

All $S$ are $P$

All $P$ are $M$

Some $S$ are not $M$

Some $S$ are not $P$

Syllogisms

Four kinds of statements:

All/Some . . . are/are not . . .

two premises, a conclusion (256 “modes”, 19 (15) valid ones).
Logic ⊆ Philosophy, Logic ⊆ Mathematics.

Circularity? Mathematics used in logic, but logic used in (or even founding?) mathematical reasoning.

Desirable properties of a formal system:
sufficiency (expressibility): Has formulas for the items that interest us.
necessity : No superfluous symbols or notions.
consistency : Two contradictory statements never concluded.
soundness : Only true statements concluded.
completeness : All true statements concluded.
decidability : Checkable if a statement is concluded or not.


<table>
<thead>
<tr>
<th>Symbols, formalism</th>
<th>Real world (objects)</th>
</tr>
</thead>
<tbody>
<tr>
<td>proof, theorem, deduction, ⊢ p</td>
<td>model, consequence, validity, ⊨ p, Alfred Tarski (1902–83) 1933</td>
</tr>
</tbody>
</table>

premises: allegedproof → statements

conclusion: allegedproof → statements

checkproof: allegedproof → bool computable!
THREE WORLDS, TWO VIEWS ON MODELING

The real world

“model” in the sense of science

World of models, meanings

“model” in the sense of logic

World of logical formulas

Engineering

\[ \mathcal{M} \models \phi \]

▶ Left side: scientific experiments, measurements, what is “out there”?

Purpose: analytic, to understand nature.

▶ Right side: specifications to define what is to be done!

Purpose: synthetic, to construct systems.
MODEL THEORY VERSUS PROOF THEORY

\[ M \models \phi \] Model (or system) \( M \) satisfies statement \( \phi \) (a formula)

- This says something about how system \( M \) behaves
- It is about validity or truth (Alfred Tarski, 1930s)

\[ \Gamma \vdash \phi \] Model \( M \) satisfies formula \( \phi \)

- This says that formula \( \phi \) can be proven from assumptions \( \Gamma \)
  (dates back to the ancient Greeks)
- It is about a formal system, not about truth

Amazingly, these two approaches are often equivalent:

One can prove: true properties about systems
by means of: formal manipulation of symbols

This is the central point of this course.
(in spite of what sometimes looks like pedantic symbol pushing.)
A major theme is the equivalence of the two:

- **Soundness**: $\vdash$ implies $\models$: What can be is true.
- **Completeness**: $\models$ implies $\vdash$: What is true can be proven.

**Propositional logic**: $\vdash$ is equivalent to $\models$, but nontrivial (in spite of the simplicity of truth tables). Reason: $SAT$ is NP-hard.

**Predicate logic**: $\vdash$ is equivalent to $\models$. However

- $\vdash$ and $\models$ are undecidable. Further,
- Gödel proved equivalence on universal models, but
- Gödel also proved that $\vdash$ is weaker than $\models$ for arithmetic (the natural numbers)

**Model checking via temporal logic**:

- Weaker than Predicate logic (formulas have no variables)
- Stronger in another way (temporal operator like “Finally”)
- There are several different temporal logics
- Designed so that $\vdash$ equals $\models$
- - and there exist efficient algorithms called model checkers
Judgements formed with propositional variables $p, q, r, p_1, \ldots$, and operators:

**Negation** $\neg \phi, \bar{\phi}, \neg \phi$

**Disjunction** Classically exclusive (lat. aut \ldots aut \ldots), now always inclusive (lat. vel \ldots vel \ldots), $\phi \lor \psi, \phi \vee \psi, \phi + \psi$

**Conjunction** $\phi \& \psi, \phi \cdot \psi, \phi \psi, \phi \land \psi$

**Implication** $\phi \supset \psi, \phi < \psi, \phi \Rightarrow \psi, \phi \rightarrow \psi$

**Absurdity, contradiction** $0, F, \phi, \Lambda, \bot$ (bottom)

Priorities: $\neg$ binds tighter than $\{\lor, \land\}$, We don’t decide on a priority between $\land$ and $\lor$.

Other logical operators:

**Exclusive disjunction** $+, \oplus$. **Equivalence** $=, \equiv, \leftrightarrow, \leftrightarrow$.

**Tautology** $1, T, \lor, \top$
A sequent $\phi_1, \ldots, \phi_n \vdash \psi$

Gerhard Gentzen (1909–45)

The rules for conjunction

\[
\frac{\phi \quad \psi}{\phi \land \psi} \quad \land_i \\
\frac{\phi \land \psi}{\phi} \quad \land_e_1 \\
\frac{\phi \land \psi}{\psi} \quad \land_e_2
\]

Example 1.4: $p \land q, r \vdash q \land r$

Proof trees

Proofs in linear form

1. $p \land q$ premise
2. $r$ premise
3. $q$ $\land_e_2$ 1
4. $q \land r$ $\land_i$ 3,2
THE RULES OF DOUBLE NEGATION

\[
\begin{align*}
\overline{\neg \phi} & \quad \overline{\neg \neg \phi} \\
\phi & \quad \phi \\
\overline{\neg \phi} & \quad \neg \phi \\
\overline{\neg \phi} & \quad \neg \phi
\end{align*}
\]

(Later we shall see that the second rule can be derived from other rules.)

**Example 1.5:** \( p, \neg \neg (q \land r) \vdash \neg \neg p \land r \)

**Example 1.6:** \((p \land q) \land r, s \land t \vdash q \land s\)
IMPLICATION

Modus ponens (MP)

\[
\begin{array}{c}
\phi \\
\phi \rightarrow \psi \\
\hline
\psi
\end{array}
\]

\[\vdash \phi \rightarrow \psi\]

\[\vdash p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r\]

Introduce \(\rightarrow:\)

\[
\begin{array}{c}
\phi \\
\vdots \\
\psi
\end{array}
\]

\[\phi \rightarrow \psi \]

Modus tollens (MT)

\[
\begin{array}{c}
\phi \rightarrow \psi \\
\neg \psi
\hline
\neg \phi
\end{array}
\]

\[\vdash \neg \phi\]

Example 1.7: \(p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q\)

Example 1.8: \(\neg p \rightarrow q, \neg q \vdash p; p \rightarrow \neg q, q \vdash \neg p\)
\[ \neg q \rightarrow \neg p \vdash p \rightarrow \neg \neg q \]

1. \( \neg q \rightarrow \neg p \) premise
2. \( p \) assumption
3. \( \neg \neg p \) \(-\neg i\ 2\)
4. \( \neg \neg q \) \(\text{MT 1,3}\)
5. \( p \rightarrow \neg \neg q \) \(\rightarrow i\ 3,2\)

Example 1.11: \( \vdash (q \rightarrow r) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)) \)

Example 1.13–1.14: \( p \land q \rightarrow r \vdash p \rightarrow (q \rightarrow r) \)

Example 1.15: \( p \rightarrow q \vdash p \land r \rightarrow q \land r \)
DISJUNCTION

\[
\begin{align*}
\frac{\phi}{\phi \lor \psi} & \quad \text{\( \lor i_1 \)} \\
\frac{\psi}{\phi \lor \psi} & \quad \text{\( \lor i_2 \)}
\end{align*}
\]

\[
\frac{
\begin{array}{cc}
\phi & \vdots \\
\psi & \vdots \\
\end{array}
}{
\phi \lor \psi & \chi \\
\chi & \chi
\end{array}
\quad \text{\( \lor e \)}
\]

\[p \lor q \vdash q \lor p\]

**Example 1.16:** \(q \rightarrow r \vdash p \lor q \rightarrow p \lor r\)

**Example 1.17:** \((p \lor q) \lor r \vdash p \lor (q \lor r)\)

**Example 1.18:** \(p \land (q \lor r) \vdash (p \land q) \lor (p \land r)\)
Example 1.20: $\neg p \lor q \vdash p \rightarrow q$

Example 1.21: $p \rightarrow q, p \rightarrow \neg q \vdash \neg p; p \rightarrow \neg p \vdash \neg p$

Example 1.22: $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$

Example 1.23: $p \land \neg q \rightarrow r, \neg r, p \vdash q
**DERIVED RULES**

\[
\begin{align*}
\phi & \to \psi \\
\quad & \quad \neg \psi \\
\quad & \quad \frac{}{\neg \phi} \quad \text{MT} \\
\quad & \quad \frac{}{\phi} \\
\quad & \quad \frac{}{\neg \neg \phi} \\
\end{align*}
\]

Reductio ad absurdum is **PBC, Proof By Contradiction**:

\[
\begin{array}{c}
\neg \phi \\
\vdots \\
\bot \\
\hline
\phi
\end{array}
\]

**PBC**

**LEM**, law of the excluded middle:

\[
\phi \lor \neg \phi
\]

**Example 1.24**: \( p \to q \vdash \neg p \lor q \)
\((p \land q) \downarrow \Downarrow \neg p \lor \neg q\)

\((p \lor q) \downarrow \Downarrow \neg p \land \neg q\)

\(p \rightarrow q \downarrow \Downarrow \neg q \rightarrow \neg p\)

\(p \rightarrow q \downarrow \Downarrow \neg p \lor q\)

\(p \land q \rightarrow p \downarrow \Downarrow r \lor \neg r\)

\(p \land q \rightarrow r \downarrow \Downarrow p \rightarrow (q \rightarrow r)\)
Luitzen Egbertus Jan Brouwer (1881–1966)

Intuitionists claim PBC, LEM, $\neg\neg e$ are invalid.

Theorem 1.26: There exist positive irrational numbers $a$ and $b$ such that $a^b$ is a rational number.

Proof (not intuitionistically valid):

Choose
1. $a = b = \sqrt{2}$, if $\sqrt{2}^\sqrt{2}$ is rational, and
2. $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ otherwise.