PREDICATE LOGIC: UNDECIDABILITY AND INCOMPLETENESS
HUTH AND RYAN 2.5, SUPPLEMENTARY NOTES 2

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- Some slides today new, some based on logic 2004 (Nils Andersen)

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- Gödel's completeness theorem
- Undecidability
  - Computability
  - Problem reduction
  - Post's correspondence problem
  - Undecidability of predicate logic
- Implications for the deductive paradigm
- Gödel's incompleteness theorem
- Implications for the deductive paradigm

SOUNDNESS AND COMPLETENESS

Let $\mathcal{M}$ be a model (interpretation) for predicate calculus formulas.

Definition Formula ϕ is valid iff $\mathcal{M} \models \phi$ holds for every model $\mathcal{M}$.

Definition

- Proof system $\vdash$ is sound iff for any closed predicate formula ϕ we have:

  $\vdash \phi$ implies $\mathcal{M} \models \phi$ for every model$\mathcal{M}$

  In brief: any provable formula is valid.

- Proof system $\vdash$ is complete iff for any closed predicate formula ϕ :

  $\vdash \phi$ if $\mathcal{M} \models \phi$ for every model$\mathcal{M}$

  In brief: any valid formula is provable.

Remark: Validity is a very strong condition to place on a formula ϕ:

ϕ must hold in all models.

In contrast to most mathematical reasoning: about one model at a time.

GÖDEL'S COMPLETENESS THEOREM

The proof system $\vdash$ described in Huth and Ryan's book is both sound and complete:

If ϕ is a closed predicate formula, then $\vdash \phi$ iff it is valid, i.e.,

$$\mathcal{M} \models \phi \text{ for every model } \mathcal{M}$$

Proof ideas (Gödel's completeness theorem):

1. **Soundness**: This is straightforward.
2. **Completeness**: This is much trickier, as it involves constructing a model of $\neg\phi$ just in case ϕ is not provable.

   Since Gödel's original proof a variety of simpler alternatives have been devised, but all are too subtle and technically involved to present here.

THE DECIDABILITY PROBLEM FOR PREDICATE LOGIC

- For a sentence ϕ, obviously either
  - ϕ is true in all models, or
  - ϕ is false in some model
- Now ϕ is satisfiable if $\mathcal{M} \models \phi$ for some model $\mathcal{M}$, so
- ϕ is unsatisfiable if and only if $\neg\phi$ is valid.

But how can we find out whether or not $\vdash \phi$ in predicate logic?

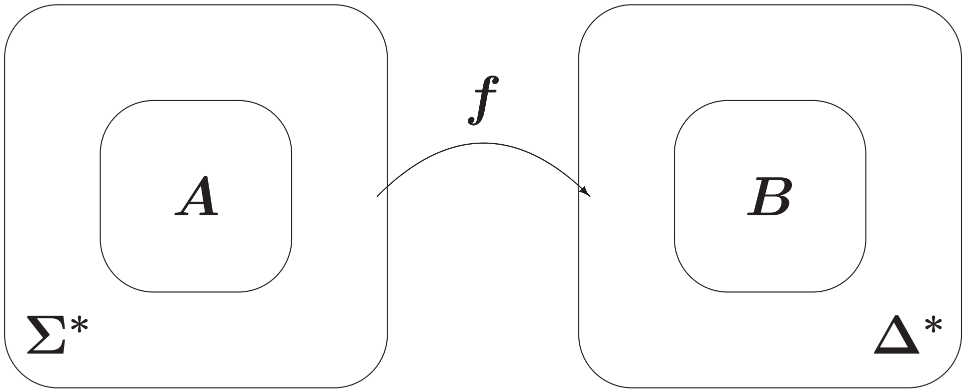
The truth-table method works for propositional logic: enumerate all possible values of propositional variables.
Alas it is no solution for predicate logic, since formula ϕ will always have infinitely many models, and so infinitely many valuations for individual variables.

Surprise: even though provability and validity are equivalent, neither one is decidable.
In other words, you cannot find out whether or not $\vdash \phi$ in predicate logic !

COMPUTABLE REDUCTION

Definition For two decision problems A over Σ and B over Δ we say that A **can be (recursively) reduced to** B, sometimes denoted $A \overset{\text{rec}}{\leq} B$, if there is a computable function $f : \Sigma^* \longrightarrow \Delta^*$ such that

$$\forall x \in \Sigma^*(x \in A \Leftrightarrow f(x) \in B)$$



Theorem Assume A can be reduced to B. If B is decidable then so is A. If A is undecidable, then so is B.

Mother of all problems unsolvable by computer: $\texttt{HALT}$, program p halts when given p as input (formally: $[\![p]\!](p)$ terminates). This is easily reduced to

$$\texttt{HALT} = \{(p, x) | [\![p]\!](x) \text{ terminates}\}$$

POST'S CORRESPONDENCE PROBLEM

Given: a finite sequence $(s_1, t_1), \ldots, (s_k, t_k)$ of pairs of strings,

To decide: is there a non-empty sequence $i_1, i_2, \ldots, i_n$ of indices such that $s_{i_1} s_{i_2} \cdots s_{i_n} = t_{i_1} t_{i_2} \cdots t_{i_n}$?

Theorem `PCP` is undecidable (proven by reducing `HALT` to it).

Without loss of generality we may assume that the strings are over a two-letter alphabet, for instance $\{0, 1\}$.

Note that `PCP` is a decision problem; but if we know the answer is positive, the actual sequence of indices can be found in an exhaustive search.

Examples of **problem instances:**

$$C = ((1, 101), (10, 00), (011, 11))$$

and the challenge

$$D = ((001, 0), (01, 011), (01, 101), (10, 001))$$

PREDICATE LOGIC IS UNDECIDABLE

Proof: Reduce PCP (over the alphabet $\{0,1\}$) to the decision problem for predicate logic. The "reduction", i.e., translation, takes an instance

$$((s_1,t_1),\ldots,(s_k,t_k))$$

to the sentence

$$\begin{array}{ll} & P(f_{s_1}(e), f_{t_1}(e)) \wedge \ldots \wedge P(f_{s_k}(e), f_{t_k}(e)) \\ \wedge & \forall v \forall w \\ & (P(v,w) \longrightarrow P(f_{s_1}(v), f_{t_1}(w)) \wedge \ldots \wedge P(f_{s_k}(v), f_{t_k}(w))) \\ \longrightarrow & \exists z P(z,z) \end{array}$$

where $f_{b_0 b_1 \ldots b_\ell}(x)$ abbreviates

$$f_{b_\ell}(\ldots(f_{b_1}(f_{b_0}(x)))\ldots)$$

(This formula uses a binary predicate symbol P, constant e and two unary function symbols f_0 and f_1.)

IMPLICATIONS FOR THE DEDUCTIVE PARADIGM

We know: If ϕ is a closed predicate formula, then

$$\vdash \phi \text{ iff } \mathcal{M} \models \phi \text{ for every model } \mathcal{M}.$$

We also know: the question $\vdash \phi$ is undecidable.

Consequence:

There exists no perfect theorem-proving system for predicate logic.

- The set of true (valid) formulas can be enumerated: just enumerate all possible proofs using, for example, Huth and Ryan's proof system.
- **But...** there exists no way to enumerate the set of false formulas (else truth would be decidable).

GÖDEL'S INCOMPLETENESS THEOREM

We now know:

1. If ϕ is a closed predicate formula, then

$$\vdash \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for every model } \mathcal{M}.$$

2. The question $\vdash \phi$ is undecidable.

Why look at all models? In practice, we're more likely interested in a specific model, e.g., the natural numbers (or reals or. . .). Alas, the situation gets even worse:

Gödel's incompleteness theorem. For

- any model $\mathcal{M}$ that (in a natural sense) includes the natural numbers, and
- for any logic system $\vdash_M$ that is sound for $\mathcal{M}$

there will exist formulas ϕ such that

ϕ is **true** in $\mathcal{M}$ but ϕ is **not provable** by $\vdash_M$.

GÖDEL NUMBERING

Towards an explicit unprovable statement.

Consider this axiomatization N of $I\!N$:

N1. $\neg S(x) = 0$

N2. $s(x) = S(y) \rightarrow x = y$

N3. $x + 0 = x$

N4. $x + S(y) = S(x + y)$

N5. $x \cdot 0 = 0$

N6. $x \cdot S(y) = (x \cdot y) + x$

N7. $\neg x < 0$

N8. $x < S(y) \leftrightarrow x < y \vee x = y$

N9. $x < y \vee x = y \vee y < x$

(Gödel used a slightly stronger system with the induction axiom schema

$$\phi[0/x] \rightarrow (\forall x(\phi \rightarrow \phi[S(x)/x]) \rightarrow \forall x \phi)$$

for every formula ϕ.)

Every term, formula and proof can be encoded as a number, its so called Gödel number. (Represent numbers $0, 1, 2, \ldots$ as numerals $0, S(0), S(S(0)), \ldots$

GÖDEL'S CONSTRUCTION

Via Gödel numbering, properties of and relations between terms, formulas and proofs become number theoretic predicates. In particular, we may consider the contrived predicate

> $W_1(u, y)$ holding if and only if u is the Gödel number of a formula ϕ with x as a free variable, and y is the Gödel number of a proof of $\phi[S^u(0)/x]$.

W_1 is decidable. It is possible to show that the system N is sufficiently strong to represent W_1, meaning:

There is a representing formula $\mathcal{W}_1$ with free variables x and y such that for all $a, b \in I\!N$, if $W_1(a, b)$ holds, then

$$N \vdash \mathcal{W}_1[S^a(0)/x][S^b(0)/y]$$

and if $W_1(a, b)$ does not hold, then

$$N \vdash \neg\mathcal{W}_1[S^a(0)/x][S^b(0)/y]$$

AN UNDECIDED STATEMENT

Now let m denote the Gödel number of the formula $\forall y \neg \mathcal{W}_1(x, y)$, and let $\mathcal{W}$ denote $\forall y \neg \mathcal{W}_1[S^m(0)/x]$.

Theorem (Gödel 1931) If N is consistent, then $\mathcal{W}$ is true in $I\!N$, but $N \nvdash \mathcal{W}$.

Proof: Assume $N \vdash \mathcal{W}$, and let k be the Gödel number of a proof of $\mathcal{W}$. Then $W_1(m, k)$, so
$N \vdash \mathcal{W}_1[S^m(0)/x][S^k(0)/y]$, so N is inconsistent.

$\mathcal{W}$ may be said to express its own unprovability.

Modifiying the construction a little, J. Barkley Rosser exhibited a formula $\mathcal{R}$ such that neither $N \vdash \mathcal{R}$ nor $N \vdash \neg\mathcal{R}$ (provided N is consistent).

These constructions can also be carried through for any extension of N.

RECURSIVE ENUMERABILITY

A set is recursively enumerable (or semi-decidable) if it is the domain of a partial computable function, i.e. if there is a program p such that x is in the set if and only if $[\![p]\!](x)$ terminates.
Equivalent: A set is recursively enumerable if it is the range of a computable function, i.e., it can be enumerated (even though it may be decidable.)

Theorem A set is decidable if and only if it itself as well as its complement are recursively enumerable.

We may conclude that the set of formulas ϕ such that $N \not\models \phi$ is not even recursively enumerable.

For any reasonable concept of provability, the theorems (the provable formulas) must be recursively enumerable.

But: $HALT = \{(p, x) | [\![p]\!](x)$ does not terminate$\}$ is not recursively enumerable.

And: $Provable = \{\phi \mid \ \vdash \phi\}$ is not recursively enumerable for any sound proof system containing the natural numbers.

IMPLICATIONS FOR THE DEDUCTIVE PARADIGM

These results have serious implications whenever we want to apply the deductive paradigm to a mathematical theory **sufficiently strong to reason about the natural numbers**.

- The set of true statements is undecidable (so nominating each of them as an axiom goes against the deductive principles).
- Selecting a decidable subset of true statements as the axioms, on the other hand, yields an inherently incomplete system: there will

  always exist true statements that cannot be proven